ON DECIDABLE, FINITELY AXIOMATIZABLE, MODAL AND TENSE LOGICS WITHOUT THE FINITE MODEL PROPERTY

PART II

BY

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ABSTRACT

This paper is a continuation of Part I. A decidable extension of model K4 that lacks the finite model property is described. It is not known whether there are extensions of B that are decidable and lack the finite model property.

Introduction

In [4] D. Makinson gave an example of a normal, finitely axiomatizable extension of T which lacks the finite model property. In [1] using methods of [3] we constructed an extension of the system of Makinson which was decidable, finitely axiomatizable, normal, and lacked the finite model property. In the proofs of the above examples an essential use was made of the fact that the systems contain $\Box\phi \rightarrow \phi$ and fail to contain $\Box\phi \rightarrow \Box\Box\phi$.

In this paper we give an example of a decidable, finitely axiomatizable extension of K4 without the finite model property. In the proof we shall make essential use of the fact that our system contains $\Box\phi \rightarrow \Box\Box\phi$ and fails to contain $\Box\phi \rightarrow \phi$.

Note that any finitely axiomatizable normal extension of K4 is also finitely axiomatizable with modus ponens as the only rule. (We always include substitution).

1. The system D_*

The following is an axiomatization of K4.

(1) All truth functional tautologies.

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(2) $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$.

(3) $\Box\phi \Rightarrow \Box\Box\phi$.

(4) Modus ponens and substitution.

(5) $\vdash \phi \to \vdash \Box\phi$.

Since we have axiom (3) we can get K4 also by dropping out (5) and adding (6) and (7) below.

(6) $\Box(\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi))$.

(7) $\Box(\Box\phi \to \Box\Box\phi)$.

We may need to axiomatize the classical propositional calculus and add (8).

(8) $\psi \wedge \Box\psi$ for ψ axiom of the classical propositional calculus.

THEOREM 9. K4 *may be finitely axiomatized with modus ponens (along with substitution).*

We now define D_*.

To get D_* add (10) to any axiomatization of K4 with modus ponens and substitution as the only rules of inference:

(10) $\Box(\Box^3\phi \to \Box^2\phi) \to \Box(\Box^2\phi \to \Box\phi)$.

It is easier to regard D_* in the light of the following.

THEOREM 11. *Let* Δ_0 *be the set of all substitution instances of schema* (10), *then*: Δ *is a complete* D_* *theory if and only if* Δ *is a complete* K4 *theory such that* $\Delta \supseteq \Delta_0$.

(11) holds for any other extension of K4 which does not have necessitation provided we take Δ_0 to be the set of all substitution instances of the additional axioms.

LEMMA 12. $\Box(\Box^2\phi \to \Box\phi)$ *is not a theorem of* D_*.

PROOF. Consider the following set of possible worlds:

$$\begin{array}{c} \vdots \\ 2 \\ 1 \\ 0 \\ -1 \\ -2 \\ \vdots \\ -\infty \end{array}$$

where $-\infty Rx$ for every $x \neq -\infty$ ($-\infty$ being our base structure) and xRy iff $x \underset{\neq}{<} y$ for finite x and y. The semantics has no special intuitive meaning except that it serves to prove (12).

We now show that (10) holds at $-\infty$. Assume that

$$[\Box(\Box^2\phi \to \Box\phi)]_{-\infty} = F$$

so at some m

$$[\Box^2\phi \to \Box\phi]_m = F$$

and so $[\Box^3\phi \to \Box^2\phi]_{m-1} = F$ and so

$$[\Box(\Box^3\phi \to \Box^2\phi)]_{-\infty} = F.$$

We thus conclude that this is a model of all the theorems of D_*.

Now to show that our sentence is not provable, let $[p]_m = T$ iff (def)$m \geqq 0$. Thus $[\Box^2 p \to \Box p]_{-2} = F$ and so $[\Box(\Box^2 \to p\ \Box p)]_{-\infty} = F$.

Lemma 13. *Any model of D_* in which $\Box(\Box^2\phi \to \Box\phi)$ is false is not finite.*

Proof. If our sentence is false in the model, then it is false at the base point 0.

$$[\Box(\Box^2\phi \to \Box\phi)]_0 = F.$$

Therefore there exists a y_1 such that $0Ry$ and $[\Box^2\phi]_{y_1} = T$ and $[\Box\phi]_{y_1} = F$.

Assume by induction that there exist $y_1 \cdots y_n$ such that $0Ry_1, \cdots, 0Ry_n$ and that for each $1 \leqq i \leqq n$ we have

(14) $$[\Box^{i+1}\phi]_{y_i} = T,\ [\Box^i\phi]_{y_i} = F.$$

In particular, we have that

(15) $$[\Box^{n+1}\phi \to \Box^n\phi]_y = F \text{ and } 0Ry_n.$$

Therefore by (10) for $\psi = \Box^{n-1}\phi$ we have $[\Box\ \Box^{n+2}\phi \to \Box^{n+1}\phi)]_0 = F$ and therefore for some y_{n+1} such that $0Ry_{n+1}$ we have $[\Box^{n+2}\phi]_{y_{n+1}} = T$ and $[\Box^{n+1}\phi]_{y_{n+1}} = F$.

(16) We claim now that all y_m, $m \in \omega$ are different.This is so since $\Box\phi \to \Box\Box\phi$ holds. So if $[\Box^m\phi]_z = T$ then $[\Box^k\phi]_z = T$ for all $k \geqq m$.

Thus we conclude the proof of (13).

Problem 17. Find semantics for D_*.

2. The system $D_\#$

To get the system $D_\#$ extend any axiomatization of (9) with

(18) $\diamond([\Box\psi \wedge \alpha \wedge \beta \wedge \Box\beta) \to \diamond(\Box^2\psi \wedge \diamond\alpha \wedge \Box\beta)$

we know that (11) applies to $D_{\#}$ as well.

We now describe a semantics with respect to which $D_{\#}$ is complete. Our models have a set $(S, <)$ of possible worlds ($<$ is our R) with the following properties (0 is base!).

(19) $<$ is transitive and not necessarily reflexive with 0 the first element.

(20) $$\forall y \exists z(0 < z < y \wedge \forall u(z < u \to y \leqq u)).$$

LEMMA 21. *All theorems of $D_{\#}$ hold in this semantics.*

PROOF. This is clearly a model of K4. All we have to verify is that (18) holds in the base.

Let $[\diamond(\Box x \wedge \alpha \wedge \beta \wedge \Box\beta)]_0 = T$. Then at some y such that $0 < y$ we have $[\Box x \wedge \alpha \wedge \beta \wedge \Box\beta]_t = T$.

Let z be the point given by (20) and so $[\Box^2\psi \wedge \diamond \alpha \wedge \Box\beta]_z = T$ and so $[\diamond(\Box^2\psi \wedge \diamond\alpha \wedge \Box\beta)]_0 = T$.

THEOREM 22. *$D_{\#}$ is complete for this semantics.*

PROOF. Let Δ be a complete $D_{\#}$ theory. By (11) we can regard Δ as a complete K4 theory such that every substitution instance of (18) is in Δ. Now we may continue as in the completeness proof for K4. Let S be the set of *all* K4 complete theories such that

(23) $$\Delta \in S.$$

(24) Whenever $\Theta \in S$ and $\sim\Box\psi \in \Theta$ then for some $\Theta^\psi \in S$ we have $\sim\psi \in \Theta^\psi$ and for all $\alpha, \Box\alpha \in \Theta \to \alpha \in \Theta^\psi$.

Define

(25) $\Theta < \Theta'$ iff (def) for all α $\Box\alpha, \in \Theta \to \alpha \in \Theta'$.

(26) Now it is known that $<$ is transitive and that if we define $[p]_\theta = T$ iff (def). $p \in \Theta$ for any propositional variable p and any Θ we get for all ϕ, $\phi \in \Theta$ iff $[\phi]_\Theta = T$ and thus $(S, <, \Delta)$ is a K4-model of Δ.

All this is well known [4]. We will now show that $(S, <, \Delta)$ fulfills the conditions of our semantics ((19), (20)).

We must be careful since we do not have necessitation. In K4 we do have necessitation and so the construction (23)-(27) can be carried out.

LEMMA 28. *Let* $\Delta < \Theta$, *then the following set is* K4 *consistent*:

$$\{\delta \mid \Box\delta \in \Delta\} \bigcup \{\Diamond\alpha \mid \alpha \in \Theta\} \bigcup \{\Box^2\psi \mid \Box\psi \in \Theta\} \bigcup \{\Box\beta \mid \beta \wedge \Box\beta \in \Theta\}.$$

PROOF. Assume otherwise then

$$\mathrm{K4} \vdash \delta_1 \wedge \cdots \wedge \delta_n \rightarrow \sim(\Diamond\alpha_1 \wedge \cdots \wedge \Diamond\alpha_k \wedge \Box^2\psi_1 \wedge \cdots \wedge \Box^2\psi_m \wedge \Box\beta_1 \wedge \cdots \wedge \Box\beta_p)$$

$$\mathrm{K4} \vdash \bigwedge\delta_i \rightarrow (\Box^2 \bigwedge\psi_i \wedge \Box \bigwedge\beta_i \rightarrow \bigvee \sim\Diamond\alpha_i)$$

$$\mathrm{K4} \vdash \bigwedge\delta_i \rightarrow (\Box^2 \bigwedge\psi_i \wedge \Box \bigwedge\beta_i \rightarrow \Box \sim \bigwedge\alpha_i).$$

We now claim that

(29) $\Box(\Box^2 \bigwedge\psi_i \wedge \Box \bigwedge\beta_i \rightarrow \Box \sim \bigwedge\alpha_i) \in \Delta.$

This is true since we have necessitation in K4 and so

$$\mathrm{K4} \vdash \Box \bigwedge\delta_i \rightarrow \Box(\Box^2 \bigwedge\psi_i \wedge \Box \bigwedge\beta_i \rightarrow \Box \sim \bigwedge\alpha_i)$$

and since Δ is a complete K4 theory and $\Box \bigwedge \delta_i \in \Delta$ we get our result.
Now call $\psi = \bigwedge\psi_i$, $\beta = \bigwedge\beta_i$, and $\alpha = \bigwedge\alpha_i$. So we have that $\beta \wedge \Box\beta \wedge \Box\psi \wedge \alpha \in \Theta$ and therefore since $\Delta < \Theta$ we get that

$$\Diamond(\Box\beta \wedge \beta \wedge \Box\psi \wedge \alpha) \in \Delta.$$

But Δ is not an ordinary K4 theory but one which contains every substitution instance of (18) and so

$$\Diamond(\Box\beta \wedge \Box^2\psi \wedge \Diamond\alpha) \in \Delta$$

or in other words

$$\sim\Box\sim(\Box\beta \wedge \Box^2\psi \wedge \Diamond\alpha) \in \Delta$$

or

$$\sim\Box(\Box\beta \wedge \Box^2\psi \rightarrow \sim\Diamond\alpha) \in \Delta$$

which contradicts 29.

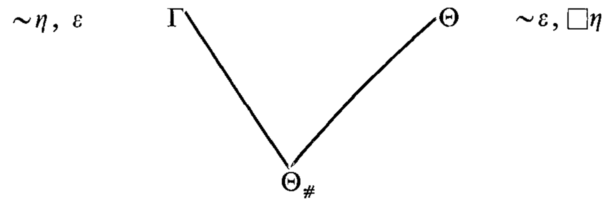
Now to continue the completeness proof, extend the set of (28) to a complete K4 theory $\Theta_\#$. By the definition of S, $\Theta_\# \in S$.

LEMMA 30.

(32) $\Delta < \Theta_\# < \Theta$

(32) $\Theta_\# < \Gamma \rightarrow \Theta = \Gamma$ or $\Theta < \Gamma$.

PROOF. (31) holds since by (28) all $\Diamond\alpha$ for $\alpha \in \Theta$ and all δ for $\Box\delta \in \Delta$ are in $\Theta_{\#}$. Now to prove (32), assume that $\Theta_{\#} < \Gamma$, $\Gamma \neq \Theta$ and $\Theta \nless \Gamma$.



So for some ε, η we have $\varepsilon \in \Gamma$, $\sim\varepsilon \in \Theta$, $\Box\eta \in \Theta$, $\sim\eta \in \Gamma$. Now consider $\beta = \eta \vee \sim\varepsilon$. We have $\Box\beta \wedge \beta \in \Theta$, so by construction we must have that $\Box\beta \in \Theta_{\#}$ and so $\beta \in \Gamma$ since $\Theta_{\#} < \Gamma$, but this is a contradiction.

Thus $(S, <, \Delta)$ fulfills our requirements and the completeness proof is concluded.

LEMMA 33. *$\Box(\Box^2\phi \rightarrow \Box\phi)$ is not a theorem of $D_{\#}$.*

PROOF. The model given in (12) is also a model of $D_{\#}$.

COROLLARY 34. *$D_{\#}$ lacks the finite model property.*

3. Decidability of $D_{\#}$

To show that $D_{\#}$ is decidable we shall use a theorem of M. O. Rabin [2], [5]. We assume familiarity with [1], in particular with the construction following Lemma 26 and with §4.

Our first step is to give a tree semantics to $D_{\#}$; we shall do this by performing the construction in the completeness proof of the last section more carefully.

Now let Δ be a complete K4 theory which contains every substitution instance of $D_{\#}$. We shall construct a tree of theories.

Stage 0. Let Δ stand at the base of the tree. Denote the base by 0 and write $\Theta(0) = \Delta$. In the later stages of the construction x, y will range over elements of the tree and $\Theta(x)$, $\Theta(y)$ will denote the theories standing at a point x or y of the tree.

We use successor functions $s_0 s_1 \cdots$ and one additional successor $r(x)$.

Stage 1. For every $\sim\Box\psi \in \Delta$ construct Δ^{ψ} (as in 24) and make them $s_0 s_1 \cdots$ successors of 0. Thus $\Theta(s_m(0))$ is some theory Δ^{ψ}.

Stage 2. For every $\Theta(x)$ constructed in stage 1 form all $\Theta(x)^{\psi}$ and make them $s_0 s_1 \cdots$ successors of $\Theta(x)$. In addition to this since we have $\Delta < \Theta(x)$, we know that $\Theta(x)_\#$ exists (by (28)) and that $\Delta < \Theta(x)_\# < \Theta(x)$ by (31) and (32). We thus may take $\Theta(x)_\#$ and make it the $r(x)$ successor of x, and so $\Theta(r(x)) = (\Theta(x))_\#$. Furthermore by (28) again we have that $\Theta(x)_{\#\#}$ exists (since $\Delta < (\Theta(x))_\#$ holds) and by (32) we know that $\Theta(x)_\#$ is the only *immediate* successor of $\Theta(x)_{\#\#}$.

Stage $n+1$. For every $\Theta(x)$ constructed in stage n for x being an s_m successor for some m continue as in stage 2. For x which is an $r(y)$ construct only $\Theta(x)_\# = \Theta(r(y))_\#$ and let it stand at the point $r(x) = r(r(y))$. We now note that in the course of construction we get a tree with the following properties:

(35) 0 has no r-successors.

(36) every $x \neq 0$ has unique r successor $r(x)$ which has no s-successors.

(37) $\Theta(x) < \Theta(y)$ whenever there exists a finite set C of points such that $x \in C$ and $y \in C$ and C is linearly ordered by the transitive closure of s-successorship. This is true because always $\Theta(x) < \Theta(s_m(x))$ by construction and $D_* \vdash \Box\phi \to \Box\Box\phi$

(38) $\Theta(0) = \Delta < \Theta(r(x))$ for any x.

(39) If $\sim\Box\psi \in \Theta(s_m(x))$ then for some n, $\sim\psi \in \Theta(s_n(s_m(x)))$.

(40) Now let $\sim\Box\psi \in \Theta(r(x))$, since $\Theta(r(x))$ is $\Theta(x)_\#$ we cannot have that $\psi \wedge \Box\psi \in \Theta(x)$ since then $\Box\psi$ would be in $\Theta(r(x))$ and so either $\sim\psi \in \Theta(x)$ or $\sim\Box\psi \in \Theta(x)$.

We now define $<$ on the tree.

(41) $x < y$ iff the following:

Case a. $x \neq 0$ and is not an r-successor. Then we let $x < y$ iff y is "above" x using s_n-successorship only.

Case b. $x \neq 0$ but x is an r successor. Then $x < y$ iff we can go *back* by r-successorship to the unique first z which is *not* an r-successor and then $z < y$ in the sense of *Case a.* z is unique below x. See also (36).

Case c. $x = 0$. 0 is $<$ of any y.

Case d. x is obtained from y by taking r-successor i.e. $x = r^m(y)$.

LEMMA 42. *This definition of* $<$ *along with the semantics is expressible in the monadic tree language* ([1], [2]) *and therefore the logic thus defined is decidable.*

All we have to show now is that this tree-semantics characterizes $D_{\#}$.

LEMMA 43. *$<$ is a transitive relation on our tree that fulfills* (20).

PROOF. It is clear that $<$ is transitive (note *case d* in (41)).

To show that (20) is fulfilled, let $0 < z$, then of course $0 < r(x) < x$ by definition. And since $r(x)$ has no other successor then $r(r(x))$ by case b and d, the only y's such that $r(x) < y$ is x itself or a z such that $x < z$ by definition.

LEMMA 44. *Let p be a propositional variable, define $[p]_x = T$ iff* (def.) *$p \in \Theta(x)$, then we have for every ϕ $[\phi]_x = T$ iff $\phi \in \Theta(x)$.*

PROOF. This follows from (37), (38), (39), (40) and the definition (41) of $<$.

(45) We thus conclude that $D_{\#}$ is complete for this tree semantics.

THEOREM 46. *$D_{\#}$ is a decidable, finitely axiomatizable, extension of* K4 *which lacks the finite model property.*

PROOF. By (42) and (45).

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